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Group-invariant solutions of the (2+1)-dimensional cubic Schrödinger equation

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Abstract
We use Lie point symmetries of the (2+1)-dimensional cubic Schrödinger equation to obtain new analytic solutions in a systematic manner. We present an analysis of the reduced ODEs, and in particular show that although the original equation is not integrable they typically can belong to the class of Painlevé-type equations.

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1. Introduction

The purpose of this paper is to find new explicit group-invariant solutions of the cubic Schrödinger equation (CSE)

$$\mathrm{i}\psi_t + \Delta\psi = a_0|\psi|^2\psi, \tag{1.1}$$

where $\psi(t, x, y)$ is a complex-valued function of its arguments, a_0 is a real constant and Δ is the Laplace operator in $\mathbb{R}^2$.

Equation (1.1) plays an important role in many branches of physics. For instance, in nonlinear optics it describes the interaction of electromagnetic waves with different polarizations propagating in nonlinear media such as an isotropic plasma. Another instance where equation (1.1) arises is water waves in a deep fluid.

In one space dimension, the CSE is known to be integrable via inverse scattering transformation. In two space dimensions this approach does not work. Therefore, our approach will rely on the use of powerful methods of symmetries of differential equations. The Lie group G of local point transformations leaving the equation invariant is well-known in the literature. We shall use subgroups of G to reduce the equation to a system of algebraic or (second-order) ordinary differential equations. At this stage, it is of vital importance to perform a classification of subalgebras. The reduced equations will be compatible with specific types of boundary conditions, such as translational invariance or a cylindrical symmetry. Then, they will be solved whenever possible, in particular when they have the Painlevé property. This approach is basically similar to that of a generalized nonlinear Schrödinger equation involving cubic, quintic equations as special cases in three space dimensions ($n = 3$) studied in a series of papers [1–3]. For a vector version of (1.1), a similar analysis for two space dimensions ($n = 2$) and two or three wavefunctions ($N = 2, 3$) is given in [4]. The present paper will complete the results of [4] to include the scalar case, namely one wave ($N = 1$).

We organize the paper as follows. In section 2, we review the symmetry group G, its algebra L and subalgebras of the Schrödinger equation. In section 3, we utilize subalgebras to perform reductions to second-order ordinary differential equations (ODEs). In section 4, we construct solutions of the reduced equations whenever possible and thereby invariant solutions of the original equation. In particular, we show that for specific values of group parameters, we can pick out equations having the Painlevé property among generically non-Painlevé ones. Finally, in section 5 we make some concluding remarks.

2. The symmetry group and its Lie algebra

By a symmetry group G we shall refer to the local Lie group of point transformations:

$$\begin{aligned} \tilde{t} &= T_g(t, x, y, \psi, \psi^*), & \tilde{x} &= X_g(t, x, y, \psi, \psi^*), \\ \tilde{y} &= Y_g(t, x, y, \psi, \psi^*), & \tilde{\psi} &= \Psi_g(t, x, y, \psi, \psi^*), \end{aligned} \tag{2.1}$$

such that $\tilde{\psi}(t, x, y)$ is a solution whenever $\psi(t, x, y)$ is one. Here the star denotes the complex conjugation and g is the group element.

The computation of the symmetry group is quite straightforward (for details, see for example, [5]). To this end, we look for the infinitesimal symmetries that close to form a Lie algebra under commutation. The Lie algebra L of the symmetry group G is realized by vector fields of the form

$$\mathbf{v} = \tau\partial_t + \xi\partial_x + \eta\partial_y + \varphi\partial_\psi + \varphi^*\partial_{\psi^*}. \tag{2.2}$$

The coefficients $\tau, \xi, \eta, \varphi, \varphi^*$ are functions of t, x, y, ψ, ψ^* to be determined from the invariance condition

$$\mathrm{pr}^{(2)}\mathbf{v}(E)|_{E=0} = 0, \tag{2.3}$$

where E is the equation under study and $\mathrm{pr}^{(2)}$ is the second prolongation of the vector field $\mathbf{v}$. This condition provides us a system of determining equations for the coefficients of $\mathbf{v}$. Solving this system we end up with a basis (depending on integration constants) of vector fields for the symmetry algebra L. We exponentiate them, namely calculate $\exp L$ to find the symmetry group G.

An inverse problem of determining the most general form of Schrödinger-type equations

$$\psi_t + F(x, y, t, \psi_i, \psi_i^*, \psi_{ij}, \psi_{ij}^*) = 0, \qquad i, j \in \{x, y\} \tag{2.4}$$

invariant under the Schrödinger and its subgroups was attacked in [6]. An investigation of this problem was reduced to that of classifying inequivalent realizations of the Schrödinger algebra. In particular, the CSE appears to be among those equations allowing the same Schrödinger algebra $\mathfrak{sch}(2)$ as its symmetry algebra. Hence, we simply resort to the results of [6] for recovering a suitable basis of the algebra written in phase and modulus coordinates. In passing, we mention that symmetry breaking interactions for the time-dependent Schrödinger equation were studied in [7] for $n = 1$ and in [8] for $n = 2$.

We introduce the moduli and phases of ψ by setting

$$\psi = R\,\mathrm{e}^{\mathrm{i}\phi}, \qquad 0 \leqslant R < \infty, \qquad 0 \leqslant \phi \leqslant 2\pi \tag{2.5}$$

and rewrite our equation (1.1) in terms of R and ϕ as a coupled system of real equations

$$-R\phi_t + \Delta R - R|\nabla\phi|^2 = a_0 R^3, \tag{2.6a}$$

$$R_t + R\Delta\phi + 2\nabla R \cdot \nabla\phi = 0, \tag{2.6b}$$

where R and ϕ are functions of t, x, y.

We summarize some facts on the structure of the symmetry algebra and its important subalgebras.

- The nine-dimensional Schrödinger algebra $\mathfrak{sch}(2)$ has a Levi decomposition

$$\begin{aligned}\mathfrak{sch}(2) &= \mathfrak{h}_2 \ni\!\!\!+ \{\mathrm{sl}(2,\mathbb{R}) \oplus \mathrm{so}(2)\} \\ &\sim \{P_1, P_2, B_1, B_2, E\} \ni\!\!\!+ \{T, C, D, J\}\end{aligned} \tag{2.7}$$

  with nonzero commutation relations

$$\begin{array}{lll} [P_1, B_1] = E/2, & [P_2, B_2] = E/2, & \\ [J, B_2] = -B_1, & [J, P_2] = -P_1, & \\ [J, B_1] = B_2, & [J, P_1] = P_2, & \\ [T, B_j] = P_j, & [D, B_j] = B_j, & j = 1, 2, \\ [D, P_j] = -P_j, & [C, P_j] = -B_j, & j = 1, 2, \\ [T, D] = 2T, & [T, C] = D, & [D, C] = 2C. \end{array} \tag{2.8}$$

  Here $\ni\!\!\!+$ denotes a semi-direct sum and $\mathfrak{h}_2$ is a Heisenberg algebra.
- The subalgebra $\{T, P_1, P_2, B_1, B_2, J, E\}$ generates the extended Galilei group (translations, proper Galilei transformations and constant change of phase), D and C generate dilations and nonrelativistic conformal transformations.
- A convenient basis for the standard Schrödinger algebra is given by the vector fields

$$T = \partial_t, \qquad P_1 = \partial_x, \qquad P_2 = \partial_y, \qquad E = \partial_\phi, \tag{2.9a}$$

$$B_1 = t\partial_x + x/2\partial_\phi, \qquad B_2 = t\partial_y + y/2\partial_\phi, \qquad J = -y\partial_x + x\partial_y, \tag{2.9b}$$

$$D = 2t\partial_t + x\partial_x + y\partial_y - R\partial_R, \tag{2.9c}$$

$$C = t^2\partial_t + xt\partial_x + yt\partial_y - tR\partial_R + \tfrac{1}{4}(x^2+y^2)\partial_\phi. \tag{2.9d}$$

- The transformations corresponding to the Galilei-similitude algebra are global and given by

$$\begin{aligned} &\tilde t = \mathrm{e}^{\lambda}(t - t_0), \\ &\tilde x = \mathrm{e}^{\lambda/2}\{\cos\alpha[(x-x_0) + v_1(t-t_0)] + \sin\alpha[(y-y_0) + v_2(t-t_0)]\}, \\ &\tilde y = \mathrm{e}^{\lambda/2}\{-\sin\alpha[(x-x_0) + v_1(t-t_0)] + \cos\alpha[(y-y_0) + v_2(t-t_0)]\}, \\ &\tilde R = \mathrm{e}^{-\lambda/2}R, \qquad \tilde\phi = \phi + \tfrac{1}{2}[v_1(x-x_0) + v_2(y-y_0)] + \tfrac{1}{4}\left(v_1^2+v_2^2\right)(t-t_0) + \delta, \end{aligned} \tag{2.10}$$

  where $t_0, x_0, y_0, v_1, v_2, \alpha, \lambda, \delta$ are group parameters.
- The conformal transformations generated by C are local and given by

$$\begin{aligned} &\tilde t = \frac{t}{1-pt}, \qquad \tilde x = \frac{x}{1-pt}, \qquad \tilde y = \frac{y}{1-pt}, \\ &\tilde R = (1-pt)R, \qquad \tilde\phi = \phi + \frac{p(x^2+y^2)}{4(1-pt)}, \end{aligned} \tag{2.11}$$

  provided $1 - pt \neq 0$. Above, p is the group parameter.
  The transformations (2.10) and (2.11) can be composed to obtain the transformations of the Schrödinger group.

- The CSE is also invariant under time reversal and reflection of the space coordinates:

$$\begin{array}{lllll} T_d: & t\to -t, & x\to x, & y\to y, & \psi\to\psi^*,\\ R_x: & t\to t, & x\to -x, & y\to y, & \psi\to\psi,\\ R_y: & t\to t, & x\to x, & y\to -y, & \psi\to\psi. \end{array} \tag{2.12}$$

The elements T, R_x, R_y generate a discrete finite group G_{D}.

Comment. *The conformal transformations generated by C do not generalize to higher dimensional cubic Schrödinger equations. We mention that in a higher dimensional space n, the nonlinear Schrödinger equation with a critical exponent $4/n$ [9]*

$$\mathrm{i}\psi_t + \Delta\psi = |\psi|^{4/n}\psi, \qquad x\in\mathbb{R}^n$$

is conformally invariant. In this case, the conformal generator C and the corresponding local group action on (x, t, R, ϕ) are given by

$$C = t^2\partial_t + t\sum_{j=1}^{n} x_j\partial_{x_j} - \frac{nt}{2}R\partial_R + \frac{|x|^2}{4}\partial_\phi, \qquad |x|^2 = \sum_{j=1}^{n} x_j^2, \tag{2.13}$$

$$\begin{array}{lll} \tilde{t} = \dfrac{t}{1-pt}, & \tilde{x}_j = \dfrac{x_j}{1-pt}, & j = 1,\ldots,n\\ \tilde{R} = (1-pt)^{n/2}R, & \tilde{\phi} = \phi + \dfrac{p|x|^2}{4(1-pt)}, & \end{array} \tag{2.14}$$

provided $1 - pt \neq 0$. The induced action of C on a solution $\psi_0(x, t)$ produces the transformation formula

$$\psi(x,t) = (1+pt)^{-n/2}\exp\left(\frac{\mathrm{i}p|x|^2}{4(1+pt)}\right)\psi_0\left(\frac{x}{1+pt}, \frac{t}{1+pt}\right), \qquad x\in\mathbb{R}^n,$$

which means that $\psi(x, t)$ is a solution whenever $\psi_0(x, t)$ is 1.

2.1. Subalgebras of the Schrödinger algebra $\mathfrak{sch}(2)$

To perform symmetry reduction for equation (1.1) we need a classification of low-dimensional subalgebras of the extended Schrödinger algebra $L = \mathfrak{sch}(2)$ into conjugacy classes under the action of the group of transformations G leaving equation (1.1) invariant. The group G has the structure

$$G = G_{\mathrm{D}} \ltimes G_0,$$

where $\ltimes$ denotes a semi-direct product. The invariant subgroup G_0 is the connected component of G, the Lie algebra of which is (2.7) and G_{D} is the discrete component defined by (2.12). In order to reduce equation (1.1) to an algebraic equation, an ordinary differential equation (ODE) or a partial differential equation (PDE) in two variables we need to know all subalgebras $L_0 \subset L$ of dimension $\dim L_0 = 3, 2$ and 1, respectively. Moreover, the corresponding subalgebras should not contain the element E alone as long as we are interested in invariant solutions. Otherwise, the subgroup invariants would be independent of the phase ϕ and this in turn would not define locally invertible transformations to ψ or ψ^*.

A complete list of representatives of all subalgebra classes of $\mathfrak{sch}(2)$ is given in [10].

Table 1. Two-dimensional subalgebras leading to ODEs ($\varepsilon = \pm 1$).

No	Type	Basis
$L_{2,1}$	$2A_1$	$J + aE, C + T + bE$
$L_{2,2}$		$J + aE, T$
$L_{2,3}$		$J + aE, T + \varepsilon E$
$L_{2,4}$		$J + aE, D + bE$
$L_{2,5}$		T, P_1
$L_{2,6}$		$T + \varepsilon E, P_1$
$L_{2,7}$		$T + B_1, P_2$
$L_{2,8}$		P_1, P_2
$L_{2,9}$		B_1, P_2
$L_{2,10}$		$J - \varepsilon(C + T) + aE, B_1 + \varepsilon P_2$
$L_{2,11}$	A_2	$D + aJ + bE, T$
$L_{2,12}$		$D + aE, P_1$

3. Symmetry reductions

A group-invariant solution is the one that is transformed to itself by group transformations. To find such a solution we choose a suitable canonical subgroup and require that the solution be left invariant. In this section we are interested in looking at solutions invariant under two-dimensional subalgebras since they lead to reductions to real systems of ODEs. We go through each representative subalgebra and find the corresponding reduced system and solve them whenever possible, and thereby a solution of the original PDE. One can transform these solutions by group transformations to obtain all other possible invariant solutions. Note that the ideas we employ here are described more fully in [1–3].

In table 1 we list representatives of two-dimensional subalgebras that provide reductions to ODEs. We note that we skipped those trivially acting on the coordinate space (t, x, y).

Subgroups that have generic orbits of codimension one in the coordinate space (t, x, y) and of codimension three in the total space (t, x, y, R, ϕ) will lead to reductions to ODEs.

The procedure for finding invariant solutions is to require that the solution $\psi(t, x, y)$ be invariant under each subalgebra. This invariance imposes constraints on the solution that are expressed by first-order linear PDEs. Their solutions imply that the invariant solutions should have the form

$$\psi(t, x, y) = f(t, x, y)M(\xi)\exp[\mathrm{i}(g(t, x, y) + \Phi(\xi))], \qquad M(\xi), \quad \Phi(\xi) \in \mathbb{R}, \tag{3.1}$$

where $f(t, x, y)$, $g(t, x, y)$ and $\xi(t, x, y)$ (the subgroup invariant) are explicitly known for each subalgebra. Below we give the expressions for ξ and the wave $\psi(t, x, y)$ together with the reduced equations for the functions $M(\xi)$ and $\Phi(\xi)$ in a list. Note that for all of the Abelian subalgebras, one of the coupled reduced equations, namely reduction of (2.6*b*) is directly integrable once. Using this we eliminate $\Phi(\xi)$ to obtain a second-order real ODE for $M(\xi)$. The non-Abelian cases are exceptionally difficult to handle. In the last case ($L_{2,12}$), we raise the order of the ODE to make a decoupling possible and hence have to deal with a third-order ODE. The subalgebra $L_{2,11}$ is the only case that has led to a reduction which we have not been able to succeed in treating analytically in any way.

- The subalgebra $L_{2,1}$

$$\begin{aligned} \psi(t, x, y) &= \frac{M(\xi)}{\sqrt{1 + t^2}} \exp(\mathrm{i}(a\theta + b \arctan t + t\xi/4 + \Phi(\xi))), \\ \xi &= \frac{x^2 + y^2}{1 + t^2}, \qquad \theta = \arctan \frac{y}{x}, \end{aligned} \tag{3.2}$$

$$M^2\dot{\Phi}\xi = C_0, \qquad C_0 = \text{const}, \tag{3.3a}$$

$$\ddot{M} = \left(\frac{b}{4\xi} + \frac{a^2}{4\xi^2} + \frac{1}{16}\right)M - \frac{\dot{M}}{\xi} + \frac{C_0^2}{\xi^2}M^{-3} + \frac{a_0}{4\xi}M^3. \tag{3.3b}$$

- The subalgebra $L_{2,2}$

$$\psi(t,x,y) = M(\rho)\,\mathrm{e}^{\mathrm{i}(a\theta+\Phi(\rho))}, \qquad \rho = (x^2+y^2)^{1/2},$$
$$\theta = \arctan\frac{y}{x}, \tag{3.4}$$

$$M^2\dot{\Phi}\rho = C_0, \qquad C_0 = \text{const}, \tag{3.5a}$$

$$\ddot{M} = \frac{a^2}{\rho^2}M + \frac{C_0^2}{\rho^2}M^{-3} - \frac{\dot{M}}{\rho} + a_0M^3. \tag{3.5b}$$

- The subalgebra $L_{2,3}$

$$\psi(t,x,y) = M(\rho)\,\mathrm{e}^{\mathrm{i}(a\theta+\varepsilon\ln t+\Phi(\rho))}, \qquad \rho = (x^2+y^2)^{1/2},$$
$$\theta = \arctan\frac{y}{x}, \tag{3.6}$$

$$M^2\dot{\Phi}\rho = C_0, \qquad C_0 = \text{const}, \tag{3.7a}$$

$$\ddot{M} = \left(\varepsilon + \frac{a^2}{\rho^2}\right)M + \frac{C_0^2}{\rho^2}M^{-3} - \frac{\dot{M}}{\rho} + a_0M^3. \tag{3.7b}$$

- The subalgebra $L_{2,4}$

$$\psi(t,x,y) = \frac{M(\xi)}{\sqrt{t}}\,\mathrm{e}^{\mathrm{i}(a\theta+\frac{b}{2}\ln t+\Phi(\xi))},$$
$$\xi = \frac{x^2+y^2}{t}, \qquad \theta = \arctan\frac{y}{x}, \tag{3.8}$$

$$M^2(8\dot{\Phi}-1)\xi = C_0, \qquad C_0 = \text{const}, \tag{3.9a}$$

$$\ddot{M} = \left(\frac{b}{8\xi} + \frac{a^2}{4\xi^2} - \frac{1}{64}\right)M + \frac{C_0^2}{64\xi^2}M^{-3} + \frac{a_0}{4\xi}M^3 - \frac{\dot{M}}{\xi}. \tag{3.9b}$$

- The subalgebra $L_{2,5}$

$$\psi(t,x,y) = M(y)\,\mathrm{e}^{\mathrm{i}\Phi(y)}, \tag{3.10}$$

$$M^2\dot{\Phi} = C_0, \qquad C_0 = \text{const}, \tag{3.11a}$$

$$\ddot{M} = C_0^2M^{-3} + a_0M^3. \tag{3.11b}$$

- The subalgebra $L_{2,6}$

$$\psi(t,x,y) = M(y)\,\mathrm{e}^{\mathrm{i}(\varepsilon t+\Phi(y))}, \tag{3.12}$$

$$M^2\dot{\Phi} = C_0, \qquad C_0 = \text{const}, \tag{3.13a}$$

$$\ddot{M} = \varepsilon M + C_0^2M^{-3} + a_0M^3. \tag{3.13b}$$

- The subalgebra $L_{2,7}$

$$\psi(t,x,y) = M(\xi)\,\mathrm{e}^{\mathrm{i}(\frac{xt}{2}-\frac{t^3}{6}+\Phi(\xi))}, \qquad \xi = x - \frac{t^2}{2}, \tag{3.14}$$

$$M^2\dot{\Phi} = C_0, \qquad C_0 = \text{const}, \tag{3.15a}$$

$$\ddot{M} = \frac{\xi}{2}M + C_0^2M^{-3} + a_0M^3. \tag{3.15b}$$

- The subalgebra $L_{2,8}$

$$\psi(t,x,y) = M(t)\,\mathrm{e}^{\mathrm{i}\Phi(t)}, \tag{3.16}$$

$$\Phi = -aC_0^2t + C_1, \qquad C_0, C_1 = \text{const}, \tag{3.17a}$$

$$M = C_0. \tag{3.17b}$$

- The subalgebra $L_{2,9}$

$$\psi(t,x,y) = M(t)\,\mathrm{e}^{\mathrm{i}(\frac{x^2}{4t}+\Phi(t))}, \tag{3.18}$$

$$M = C_0t^{-\frac{1}{2}}, \tag{3.19a}$$

$$\Phi = -2aC_0t^{\frac{1}{2}} + C_1, \qquad C_0, C_1 = \text{const}. \tag{3.19b}$$

- The subalgebra $L_{2,10}$

$$\begin{aligned} \psi(t,x,y) &= \frac{M(\xi)}{\sqrt{1+t^2}}\,\mathrm{e}^{\mathrm{i}\phi(t,x,y)}, \\ \phi(t,x,y) &= \frac{x^2}{4t} - a\varepsilon\arctan t + \frac{\xi^2(t^2-1)}{4t} + \Phi(\xi), \\ \xi &= \frac{\varepsilon x - yt}{1+t^2}, \end{aligned} \tag{3.20}$$

$$M^2\dot{\Phi} = C_0, \qquad C_0 = \text{const}, \tag{3.21a}$$

$$\ddot{M} = (\xi^2 - a\varepsilon)M + C_0^2M^{-3} + a_0M^3. \tag{3.21b}$$

- The subalgebra $L_{2,11}$

$$\begin{aligned} &\psi(t,x,y) = \frac{M(\xi)}{\rho}\,\mathrm{e}^{\mathrm{i}(\frac{b}{a}\theta+\Phi(\xi))}, \qquad \xi = \rho\exp(-\theta/a), \\ &\rho = (x^2+y^2)^{1/2}, \qquad \theta = \arctan\frac{y}{x}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} &(1+a^2)\xi M\ddot{\Phi} + 2(1+a^2)\xi\dot{M}\dot{\Phi} + (1-a^2)M\dot{\Phi} - 2b\dot{M} = 0, \\ &(a^2-b^2)M + (1-a^2)\xi\dot{M} + (1+a^2)\xi^2\ddot{M} + 2b\xi M\dot{\Phi} - (1+a^2)\xi^2M\dot{\Phi}^2 \\ &\qquad = a^2a_0M^3. \end{aligned} \tag{3.23}$$

- The subalgebra $L_{2,12}$

$$\psi(t,x,y) = \frac{M(\xi)}{\sqrt{t}}\,\mathrm{e}^{\mathrm{i}(\frac{a}{2}\ln t+\Phi(\xi))}, \qquad \xi = \frac{y}{\sqrt{t}}, \tag{3.24}$$

$$\dot{\Phi} = \frac{C_0 + \xi + \xi\dot{\chi}}{4\dot{\chi}}, \qquad \dot{\chi} = M^2, \qquad C_0 = \text{const}, \tag{3.25a}$$

Table 2. Coefficients of (4.1).

No	A_1	A_3	B_0	B_1
$L_{2,1}$	$b/(4\xi)+a^2/(4\xi^2)+1/16$	$a_0/(4\xi)$	C_0^2/ξ^2	$-1/\xi$
$L_{2,2}$	a^2/ρ^2	a_0	C_0^2/ρ^2	$-1/\rho$
$L_{2,3}$	$\varepsilon+a^2/\rho^2$	a_0	C_0^2/ρ^2	$-1/\rho$
$L_{2,4}$	$b/(8\xi)+a^2/(4\xi^2)-1/64$	$a_0/(4\xi)$	$C_0^2/(64\xi^2)$	$-1/\xi$
$L_{2,5}$	0	a_0	C_0^2	0
$L_{2,6}$	ε	a_0	C_0^2	0
$L_{2,7}$	$\xi/2$	a_0	C_0^2	0
$L_{2,10}$	$\xi^2-a\varepsilon$	a_0	C_0^2	0

$$2\dot{\chi}\dddot{\chi}-\ddot{\chi}^2+\left(\frac{\xi^2}{4}-2a\right)\dot{\chi}^2-\frac{1}{4}\chi^2-4a_0\dot{\chi}^3-\frac{C_0}{2}\chi-\frac{C_0^2}{4}=0. \tag{3.25b}$$

4. Discussion of the solutions of reduced ODEs

In section 3, we showed that for the Abelian subalgebras of type $2A_1$ the reduced pair of equations can always be decoupled by integrating the second one once. For the non-Abelian case of type A_2, decoupling is possible only for one subalgebra at the price of introducing third-order derivative. Luckily, it happens to have the Painlevé property and its solution is expressible in terms of the fourth Painlevé transcendent P_{IV}.

For each reduction corresponding to subalgebras $L_{2,i}$, $i=1,\ldots,10$, $M(\xi)$ satisfies a second-order ordinary differential equation. All of them can be written in a unified form (see table 2):

$$\ddot{M}=B_1(\xi)\dot{M}+A_1(\xi)M+A_3(\xi)M^3+B_0(\xi)M^{-3}. \tag{4.1}$$

We change the dependent variable to H by putting

$$M=\sqrt{H},\qquad H>0, \tag{4.2}$$

so that the equation is transformed to

$$\ddot{H}=\frac{1}{2H}\dot{H}^2+B_1\dot{H}+2A_1H+2A_3H^2+\frac{2B_0}{H}. \tag{4.3}$$

To simplify (4.3) we apply the transformation

$$H=\lambda(\xi)W(\eta),\qquad \eta=\eta(\xi) \tag{4.4}$$

and obtain

$$\ddot{W}=\frac{1}{2W}\dot{W}^2+\left(\frac{B_1}{\dot{\eta}}-\frac{\ddot{\eta}}{\dot{\eta}^2}-\frac{1}{\dot{\eta}}\frac{\dot{\lambda}}{\lambda}\right)\dot{W}+\left(\frac{2A_1}{\dot{\eta}^2}+\frac{\dot{\lambda}}{\lambda}\frac{B_1}{\dot{\eta}^2}+\frac{\dot{\lambda}^2}{2\lambda^2\dot{\eta}^2}-\frac{\ddot{\lambda}}{\lambda\dot{\eta}^2}\right)W$$
$$+\frac{2\lambda A_3}{\dot{\eta}^2}W^2+\frac{2B_0}{\lambda^2\dot{\eta}^2}W^{-1}. \tag{4.5}$$

Equation (4.5) can be transformed to one of the canonical equations classified by Painlevé and Gambier (see [11] for a list of canonical equations). We shall see that only for special values of coefficients they can be integrated in terms of solutions of linear equations, elliptic functions or Painlevé transcendents. Otherwise, they would not have the Painlevé property which means that they cannot be integrated.

4.1. Solutions using the subalgebra $L_{2,1}$

Equation (4.5) for (3.3*b*) is

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 - \frac{1}{\dot{\eta}}\left(\frac{1}{\xi} + \frac{\ddot{\eta}}{\dot{\eta}} + \frac{\dot{\lambda}}{\lambda}\right)\dot{W} + \frac{a_0\lambda}{2\xi\dot{\eta}^2}W^2 + \frac{2C_0^2}{(\xi\lambda\dot{\eta})^2}W^{-1}$$
$$+ \frac{1}{\dot{\eta}^2}\left(\frac{b}{2\xi} + \frac{a^2}{2\xi^2} + \frac{1}{8} - \frac{\dot{\lambda}}{\lambda\xi} + \frac{\dot{\lambda}^2}{2\lambda^2} - \frac{\ddot{\lambda}}{\lambda}\right)W. \tag{4.6}$$

This equation may be transformed to either PXXXIII or PXXXIV among the canonical equations [11, 12]. For completeness, we quote them here:

$$\text{PXXXIII}: \quad \ddot{\mathrm{W}} = \frac{1}{2\mathrm{W}}\dot{\mathrm{W}}^2 + 4\mathrm{W}^2 + \gamma\mathrm{W} - \frac{1}{2}\mathrm{W}^{-1}, \tag{4.7}$$

$$\text{PXXXIV}: \quad \ddot{\mathrm{W}} = \frac{1}{2\mathrm{W}}\dot{\mathrm{W}}^2 + 4\gamma\mathrm{W}^2 - \eta\mathrm{W} - \frac{1}{2}\mathrm{W}^{-1}. \tag{4.8}$$

Comparing the coefficients of W^2 and $\dot{W}$ in (4.6) with (4.7) and (4.8) we see that λ and η must be chosen as

$$\lambda = \lambda_0\xi^{-1/3}, \qquad \eta = \eta_0\xi^{1/3}. \tag{4.9}$$

For this choice of λ and η (4.6) becomes

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + \frac{9a_0\lambda_0}{2\eta_0^2}W^2 + 18\left(\frac{C_0}{\lambda_0\eta_0}\right)^2 W^{-1} + \left[\frac{(9a^2-1)}{2}\eta^{-2} + \frac{9b}{2\eta_0^3}\eta + \frac{9}{8\eta_0^6}\eta^4\right]W. \tag{4.10}$$

Inspecting (4.7) and (4.8), we see that the coefficient of W must be a constant or η. For equation (4.10), in the coefficient of W, it is possible to remove the term η^{-2} by choosing $a^2 = 1/9$ but we cannot get rid of the term η^4. Hence equation (3.3*b*) cannot be transformed to one of the canonical forms; thus it does not have the Painlevé property.

If we consider equation (3.3*b*) with $C_0 = 0$, similar arguments apply. We mention that equation (3.3*b*) admits no point symmetries and no constant solutions.

4.2. Solutions using the subalgebra $L_{2,2}$

4.2.1. Equation (3.5B), with $C_0 \neq 0$. By a suitable choice of λ and η this equation can be transformed to a standard form which has no first derivative and a constant coefficient of the quadratic term. The transformation and the corresponding transformed equation are given by

$$\lambda = \lambda_0\rho^{-2/3}, \qquad \eta = \eta_0\rho^{2/3},$$
$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + \left(\frac{9a_0\lambda_0}{2\eta_0^2}\right)W^2 + \left(\frac{9a^2-1}{2}\eta^{-2}\right)W + \frac{9}{2}\left(\frac{C_0}{\lambda_0\eta_0}\right)^2 W^{-1}.$$

The above equation has the Painlevé property only when $a^2 = 1/9$. In this case we set $\eta_0^2 = 9a_0\lambda_0/8$ and integrate once to obtain

$$\dot{W}^2 = 4(W^3 + CW - \delta^2), \qquad C = \text{const}, \qquad \delta^2 = \frac{2C_0^2}{a_0\lambda_0^3}, \tag{4.11}$$

whose solution can be expressed in terms of Weierstrass $\wp(\xi)$ function or Jacobi elliptic functions [13]. By virtue of (4.2) we require λW to be positive. Writing the right-hand side of (4.11) as

$$P(W) = 4(W^3 + CW - \delta^2) = 4(W - W_1)(W - W_2)(W - W_3),$$

where W_1, W_2, W_3 are roots of the cubic polynomial $P(W)$, we investigate the solutions of (4.11) for four different cases:

(i) $P(W)$ has only one root. This is not possible since $P(W)$ does not include the term W^2.

(ii) $P(W)$ has a double root, say $W_1 = W_2 \neq W_3$ and write $P(W)$ as

$$P(W) = 4(W^3 + CW - \delta^2) = 4(W - W_1)^2(W - W_3).$$

This case is possible for $C = -3.2^{\frac{-2}{3}}\delta^{\frac{4}{3}}$ and $W_{1,2,3}$ are given as $W_{1,2} = -2^{-\frac{1}{3}}\delta^{\frac{2}{3}}$, $W_3 = (2\delta)^{\frac{2}{3}}$. Integrating for $W > W_3$ we reach the solution given below:

$$\begin{aligned} &M = \frac{2c_1}{3}\left(\frac{2}{a_0}\right)^{\frac{1}{2}}\rho^{-\frac{1}{3}}\left(-\frac{1}{3} + \sec^2\tau\right)^{\frac{1}{2}}, \qquad \tau = c_1\rho^{\frac{2}{3}} + c_2, \qquad c_1^2 = \frac{27}{8}(a_0C_0)^{\frac{2}{3}} \\ &\Phi = \tan^{-1}\left(\sqrt{\frac{3}{2}}\tan\tau\right) - \sqrt{\frac{2}{3}}\tau + \Phi_0, \qquad \psi = M\exp\left[\mathrm{i}\left(\pm\frac{1}{3}\theta + \Phi\right)\right]. \end{aligned} \tag{4.12}$$

(iii) $P(W)$ has three distinct roots. In this case, roots $W_{1,2,3}$ are determined by the system

$$W_1 + W_2 + W_3 = 0, \qquad W_1W_2 + W_1W_3 + W_2W_3 = C, \qquad W_1W_2W_3 = \delta^2.$$

We see that two of the roots have the same sign, while the third is always positive. Their solutions depend on ordering of the roots:

$$\text{(a)}\quad W_1 < W_2 < 0 < W_3, \qquad \text{(b)}\quad 0 < W_1 < W_2 < W_3, \qquad W_2^2 < \frac{\delta^2}{W_1}.$$

Integration for $W > W_3$ gives for both (a) and (b)

$$\begin{aligned} &W = W_2 + (W_3 - W_2)[\mathrm{cn}(\tau, k)]^{-2}, \qquad \tau = c_1\rho^{\frac{2}{3}} + c_2, \qquad k^2 = \frac{W_2 - W_1}{W_3 - W_1}, \\ &M = \rho^{-1/3}(\lambda_0 W)^{1/2}, \qquad c_1^2 = \frac{9a_0\lambda_0}{8}(W_3 - W_1), \\ &\Phi = \frac{\delta}{\sqrt{W_3 - W_1}}\int\frac{\mathrm{d}\tau}{W} + \Phi_0, \qquad \psi = M\exp\left[\mathrm{i}\left(\pm\frac{1}{3}\theta + \Phi\right)\right] \end{aligned} \tag{4.13}$$

with $a_0, \lambda_0 > 0$. Integrating on $W_1 < W < W_2$ we get

$$\begin{aligned} &W = W_1 + (W_2 - W_1)\,\mathrm{sn}^2(\tau, k), \qquad \tau = c_1\rho^{\frac{2}{3}} + c_2, \qquad k^2 = \frac{W_2 - W_1}{W_3 - W_1} \\ &M = \rho^{-1/3}(\lambda_0 W)^{1/2}, \qquad c_1^2 = \frac{9a_0\lambda_0}{8}(W_3 - W_1) \\ &\Phi = \frac{\delta}{\sqrt{W_3 - W_1}}\int\frac{\mathrm{d}\tau}{W} + \Phi_0, \qquad \psi = M\exp\left[\mathrm{i}\left(\pm\frac{1}{3}\theta + \Phi\right)\right] \end{aligned} \tag{4.14}$$

with $a_0, \lambda_0 < 0$ for the case (a) and $a_0, \lambda_0 > 0$ for the case (b).

(iv) Last we investigate the case where $P(W)$ has two complex roots and a real root. Let $W_2 = p + \mathrm{i}q$, $\mathrm{W}_3 = \mathrm{p} - \mathrm{i}q$, then p, q and W_1 are found from the system of equations

$$2p + W_1 = 0, \qquad \frac{\delta^2}{W_1} + 2pW_1 = C, \qquad p^2 + q^2 = \frac{\delta^2}{W_1}.$$

Since $W_1 > 0$, integration on $W > W_1$ provides the following solution for $a_0, \lambda_0 > 0$

$$\begin{aligned} W &= \frac{A + W_1 + (W_1 - A)\,\mathrm{cn}(\tau, k)}{1 + \mathrm{cn}(\tau, k)}, & \tau &= c_1\rho^{\frac{2}{3}} + c_2, \\ M &= \rho^{-1/3}(\lambda_0 W)^{1/2}, & k^2 &= \frac{A + p - W_1}{2A}, \\ \Phi &= \frac{\delta}{2\sqrt{A}} \int \frac{\mathrm{d}\tau}{W} + \Phi_0, & c_1^2 &= \frac{3Aa_0\lambda_0}{2}, \\ \psi &= M \exp\left[\mathrm{i}\left(\pm\frac{1}{3}\theta + \Phi\right)\right], & A^2 &= (p - W_1)^2 + q^2. \end{aligned} \tag{4.15}$$

4.2.2. Equation (3.5B), with $C_0 = 0$. From (4.11) We have

$$\dot{W}^2 = 4W(C + W^2), \qquad C = \text{const.} \tag{4.16}$$

Depending on the integration constant C, we obtain the following solutions for (1.1):

(i) $C = 0$. In this case the solution is written as

$$\psi = \frac{2}{3}\left(\frac{2}{a_0}\right)^{1/2}(\rho^{2/3} - \rho_0)^{-1}\exp\left[\mathrm{i}\left(\pm\frac{1}{3}\theta + \Phi_0\right)\right], \qquad \Phi_0 = \text{const.} \tag{4.17}$$

(ii) $C = -p^2 < 0$. We write equation (4.16) as

$$\dot{W}^2 = 4W(W - p)(W + p).$$

Integration on $W > p$ and $-p < W < 0$ gives respectively

$$\begin{aligned} \psi &= \frac{2c_1}{3}\left(\frac{1}{a_0}\right)^{1/2}\rho^{-1/3}[\mathrm{cn}(c_1\rho^{2/3} + c_2, k)]^{-1}\exp\left[\mathrm{i}\left(\pm\frac{1}{3}\theta + \Phi_0\right)\right], \qquad a_0, \lambda_0 > 0; \\ \psi &= \frac{2c_1}{3}\left(\frac{-1}{a_0}\right)^{1/2}\rho^{-1/3}\,\mathrm{cn}\left(c_1\rho^{2/3} + c_2, k\right)\exp\left[\mathrm{i}\left(\pm\frac{1}{3}\theta + \Phi_0\right)\right], \qquad a_0, \lambda_0 < 0 \end{aligned} \tag{4.18}$$

with $c_1^2 = \frac{9a_0\lambda_0(-C)^{1/2}}{4}$, $k^2 = 1/2$.

(iii) $C = p^2 > 0$. The solution for $a_0, \lambda_0 > 0$ is given by integrating (4.16)

$$\psi = \frac{2c_1}{3}\left(\frac{1}{a_0}\right)^{1/2}\rho^{-1/3}\,\mathrm{tn}(c_1\rho^{2/3} + c_2, k)\,\mathrm{dn}(c_1\rho^{2/3} + c_2, k)\exp\left[\mathrm{i}\left(\pm\frac{1}{3}\theta + \Phi_0\right)\right] \tag{4.19}$$

with $c_1^2 = \frac{9a_0\lambda_0 C^{1/2}}{8}$, $k^2 = 1/2$.

4.3. Solutions using the subalgebra $L_{2,3}$

(i) Equation (3.7*b*) with $C_0 \neq 0$. If we set $\lambda = \lambda_0\rho^{-2/3}$ and $\eta = \eta_0\rho^{2/3}$ equation (3.7*b*) becomes

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + \frac{9a_0\lambda_0}{2\eta_0^2}W^2 + \left[\frac{9a^2 - 1}{2}\eta^{-2} + \frac{9\varepsilon}{2\eta_0^3}\eta\right]W + \frac{9}{2}\left(\frac{C_0}{\lambda_0\eta_0}\right)^2 W^{-1}.$$

If we choose $\eta_0 = -\left(\frac{9}{2}\right)^{1/3}\varepsilon$ and $a^2 = 1/9$, this equation transforms to an equation quite similar to PXXXIV:

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + 4\gamma W^2 - \eta W + 2\delta^2 W^{-1},$$

where $\gamma = 3^{2/3}2^{-7/3}a_0\lambda_0$, $\delta^2 = 3^{2/3}2^{-4/3}(C_0/\lambda_0)^2$. We put [11]

$$2\gamma W = \dot{V} + V^2 + \frac{\eta}{2}$$

and see that V solves

$$\ddot{V} = 2V^3 + \eta V + k, \qquad k = -\tfrac{1}{2} \pm 4a_0\delta\mathrm{i},$$

which is the second Painlevé transcendent. Hence we get $V = P_{\mathrm{II}}\big[-\big(\tfrac{9}{2}\big)^{1/3}\varepsilon\rho^{2/3}\big]$. Since W is complex-valued, λ_0 is a complex constant which must be chosen appropriately to make $\lambda_0 W$ real.

(ii) Equation (3.7*b*) with $C_0 = 0$. We have

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + \frac{9a_0\lambda_0}{2\eta_0^2}W^2 + \left[\frac{9a^2-1}{2}\eta^{-2} + \frac{9\varepsilon}{2\eta_0^3}\eta\right]W.$$

With the choice of $a^2 = \frac{1}{9}$, $\eta_0 = \big(\frac{3}{2}\big)^{2/3}\varepsilon$ and $\lambda_0 = \big(\frac{2}{9}\big)^{5/3}\frac{1}{a_0}$, this equation transforms to PXX:

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + 4W^2 + 2\eta W.$$

Setting $U^2 = W$ we again reach the Painlevé transcendent P_{II}:

$$\ddot{U} = 2U^3 + \eta U.$$

Hence we write the solution as

$$\psi = \lambda_0^{1/2}P_{\mathrm{II}}(\eta_0\rho^{2/3})\exp\left[\mathrm{i}\left(\pm\frac{1}{3}\theta + \varepsilon\ln t + \Phi_0\right)\right]. \tag{4.20}$$

4.4. Solutions using the subalgebra $L_{2,4}$

Equation (4.5) for (3.9*b*) is

$$\begin{aligned}\ddot{W} = {}& \frac{1}{2W}\dot{W}^2 - \frac{1}{\dot\eta}\left(\frac{1}{\xi} + \frac{\ddot\eta}{\dot\eta} + \frac{\dot\lambda}{\lambda}\right)\dot{W} + \frac{a_0\lambda}{2\xi\dot\eta^2}W^2 + \frac{C_0^2}{32(\xi\lambda\dot\eta)^2}W^{-1} \\ & + \frac{1}{\dot\eta^2}\left(\frac{b}{4\xi} + \frac{a^2}{2\xi^2} - \frac{1}{32} - \frac{\dot\lambda}{\lambda\xi} + \frac{\dot\lambda^2}{2\lambda^2} - \frac{\ddot\lambda}{\lambda}\right)W.\end{aligned} \tag{4.21}$$

As was done for the subalgebra $L_{2,1}$, this equation may be comparable to PXXXIII or PXXXIV of [11]. If we compare the coefficients of W^2 and $\dot{W}$ in (4.21) with (4.7) and (4.8) we see that λ and η must be chosen as

$$\lambda = \lambda_0\xi^{-1/3}, \qquad \eta = \eta_0\xi^{1/3}.$$

For this choice of λ and η, a comparison of the coefficients with those of the possible canonical equations shows us that it cannot have the Painlevé property.

If we consider equation (3.9*b*) with $C_0 = 0$, the same arguments above follow. We stress that equation (3.9*b*) admits no point symmetries and no constant solutions.

4.5. Solutions using the subalgebra $L_{2,5}$

4.5.1. Equation (3.11B), with $C_0 \neq 0$. By a suitable choice of λ and η this equation can be transformed to a standard form which has no first derivative and a constant coefficient of the quadratic term. The transformation and the corresponding transformed equation are given by

$$\lambda = \lambda_0, \qquad \eta = \eta_0 y, \qquad \ddot{W} = \frac{1}{2W}\dot{W}^2 + \left(\frac{2a_0\lambda_0}{\eta_0^2}\right)W^2 + 2\left(\frac{C_0}{\lambda_0\eta_0}\right)^2 W^{-1}.$$

In this case we set $\eta_0^2 = a_0\lambda_0/2$ and integrate once to obtain the differential equation for Weierstrass $\wp(\xi)$ function or Jacobi elliptic functions [13]:

$$\dot{W}^2 = 4(W^3 + CW - \delta^2), \qquad C = \text{const}, \qquad \delta^2 = \frac{2C_0^2}{a_0\lambda_0^3}. \tag{4.22}$$

By virtue of (4.2) we require λW to be positive. If we factor the cubic polynomial on the right-hand side

$$P(W) = 4(W^3 + CW - \delta^2) = 4(W - W_1)(W - W_2)(W - W_3),$$

where W_1, W_2, W_3 are roots of the cubic polynomial $P(W)$, we see that there are four possibilities for the solutions of (4.22):

(i) $P(W)$ has only one root. This is not possible since $P(W)$ does not include the quadratic term W^2.

(ii) $P(W)$ has a double root, say $W_1 = W_2 \neq W_3$ and write $P(W)$ as

$$P(W) = 4(W^3 + CW - \delta^2) = 4(W - W_1)^2(W - W_3).$$

This case is possible for $C = -3.2^{\frac{-2}{3}}\delta^{\frac{4}{3}}$ and $W_{1,2,3}$ are given as $W_{1,2} = -2^{-\frac{1}{3}}\delta^{\frac{2}{3}}$, $W_3 = (2\delta)^{\frac{2}{3}}$. Integrating for $W > W_3$ we arrive at the solution given by

$$\begin{aligned} &M = c_1\left(\frac{2}{a_0}\right)^{\frac{1}{2}}\left(-\frac{1}{3} + \sec^2\tau\right)^{\frac{1}{2}}, && \tau = c_1 y + c_2, \quad c_1^2 = 3.2^{-4/3}a_0\lambda_0\delta^{2/3}, \\ &\Phi = \tan^{-1}\left(\sqrt{\frac{3}{2}}\tan\tau\right) - \sqrt{\frac{2}{3}}\tau + \Phi_0, && \psi = M\exp(\mathrm{i}\Phi). \end{aligned} \tag{4.23}$$

(iii) $P(W)$ has three distinct roots. In this case, roots $W_{1,2,3}$ are determined by the system

$$W_1 + W_2 + W_3 = 0, \qquad W_1W_2 + W_1W_3 + W_2W_3 = C, \qquad W_1W_2W_3 = \delta^2.$$

We see that two of the roots have the same sign, while the third is always positive. There exist two orderings of the roots yielding different solutions:

$$\text{(a) } W_1 < W_2 < 0 < W_3 \qquad \text{(b) } 0 < W_1 < W_2 < W_3, \qquad W_2^2 < \frac{\delta^2}{W_1}.$$

Integration for $W > W_3$ gives for both (a) and (b)

$$\begin{aligned} &W = W_2 + (W_3 - W_2)[\mathrm{cn}(\tau, k)]^{-2}, && \tau = c_1 y + c_2, \qquad k^2 = \frac{W_2 - W_1}{W_3 - W_1}, \\ &M = (\lambda_0 W)^{1/2}, && c_1^2 = \frac{a_0\lambda_0}{2}(W_3 - W_1), \\ &\Phi = \frac{\delta}{\sqrt{W_3 - W_1}}\int \frac{\mathrm{d}\tau}{W} + \Phi_0, && \psi = M\exp(\mathrm{i}\Phi) \end{aligned} \tag{4.24}$$

with $a_0, \lambda_0 > 0$.

Integrating on $W_1 < W < W_2$ we get

$$\begin{aligned} &W = W_1 + (W_2 - W_1)\,\mathrm{sn}^2(\tau, k), && \tau = c_1 y + c_2, \qquad k^2 = \frac{W_2 - W_1}{W_3 - W_1}, \\ &M = (\lambda_0 W)^{1/2}, && c_1^2 = \frac{a_0\lambda_0}{2}(W_3 - W_1), \\ &\Phi = \frac{\delta}{\sqrt{W_3 - W_1}}\int \frac{\mathrm{d}\tau}{W} + \Phi_0, && \psi = M\exp(\mathrm{i}\Phi) \end{aligned} \tag{4.25}$$

with $a_0, \lambda_0 < 0$ for the case (a) and $a_0, \lambda_0 > 0$ for the case (b).

(iv) Finally, we investigate the case where $P(W)$ has two complex roots and a real root. Let $W_2 = p + \mathrm{i}q$, $W_3 = p - \mathrm{i}q$, then p, q and W_1 are found from the system of equations

$$2p + W_1 = 0, \qquad \frac{\delta^2}{W_1} + 2pW_1 = C, \qquad p^2 + q^2 = \frac{\delta^2}{W_1}.$$

Since $W_1 > 0$, integration on $W > W_1$ gives the following solution for $a_0, \lambda_0 > 0$:

$$\begin{aligned} W &= \frac{A + W_1 + (W_1 - A)\,\mathrm{cn}(\tau, k)}{1 + \mathrm{cn}(\tau, k)}, & \tau &= c_1 y + c_2, \\ M &= (\lambda_0 W)^{1/2}, & k^2 &= \frac{A + p - W_1}{2A}, \\ \Phi &= \frac{\delta}{2\sqrt{A}} \int \frac{\mathrm{d}\tau}{W} + \Phi_0, & c_1^2 &= 2Aa_0\lambda_0, \\ \psi &= M \exp(\mathrm{i}\Phi), & A^2 &= (p - W_1)^2 + q^2. \end{aligned} \tag{4.26}$$

4.5.2. Equation (3.11B), with $C_0 = 0$. From (4.22) we have

$$\dot{W}^2 = 4W(C + W^2), \qquad C = \text{const}. \tag{4.27}$$

Depending on the integration constant C, we obtain the following elliptic solutions for (1.1):

(i) $C = 0$. In this case the solution is written for $a_0 > 0$:

$$\psi = \left(\frac{2}{a_0}\right)^{1/2} (y - y_0)^{-1} \exp(\mathrm{i}\Phi_0). \tag{4.28}$$

(ii) $C = -p^2 < 0$. We write equation (4.27) as

$$\dot{W}^2 = 4W(W - p)(W + p).$$

Integration for $W > p$ and $-p < W < 0$ gives respectively

$$\begin{aligned} \psi &= c_1 \left(\frac{1}{a_0}\right)^{1/2} [\mathrm{cn}(c_1 y + c_2, k)]^{-1} \exp(\mathrm{i}\Phi_0), & a_0, \lambda_0 &> 0; \\ \psi &= c_1 \left(\frac{-1}{a_0}\right)^{1/2} \mathrm{cn}\,(c_1 y + c_2, k) \exp(\mathrm{i}\Phi_0), & a_0, \lambda_0 &< 0 \end{aligned} \tag{4.29}$$

with $c_1^2 = a_0\lambda_0(-C)^{1/2}$, $k^2 = 1/2$.

(iii) $C = p^2 > 0$. The solution for $a_0, \lambda_0 > 0$ is given by integrating (4.27):

$$\psi = c_1 \left(\frac{2}{a_0}\right)^{1/2} \mathrm{tn}(c_1 y + c_2, k)\, \mathrm{dn}(c_1 y + c_2, k) \exp(\mathrm{i}\Phi_0) \tag{4.30}$$

with $c_1^2 = \frac{a_0\lambda_0 C^{1/2}}{2}$, $k^2 = 1/2$.

4.6. Solutions using the subalgebra $L_{2,6}$

4.6.1. Equation (3.13B), with $C_0 \neq 0$. By a suitable choice of λ and η this equation can be transformed to a standard form which has no first derivative and a constant coefficient of the quadratic term. The transformation and the corresponding transformed equation are given by

$$\lambda = \lambda_0, \qquad \eta = \eta_0 y, \qquad \ddot{W} = \frac{1}{2W}\dot{W}^2 + \left(\frac{2a_0\lambda_0}{\eta_0^2}\right) W^2 + \frac{2\varepsilon}{\eta_0^2} W + 2\left(\frac{C_0}{a_0\lambda_0}\right)^2 W^{-1}.$$

In this case we set $\eta_0^2 = a_0\lambda_0/2$ and integrate once to obtain

$$\dot{W}^2 = 4(W^3 + \alpha W^2 + CW - \delta^2), \qquad C = \text{const}, \tag{4.31}$$

where $\delta^2 = 2C_0^2/\left(a_0\lambda_0^3\right)$ and $\alpha = 2\varepsilon/(a_0\lambda_0)$. Solution of this equation can be expressed in terms of Weierstrass $\wp(\xi)$ function or Jacobi elliptic functions. By virtue of (4.2) we require λW to be positive. Let us write the right-hand side as

$$P(W) = 4(W^3 + \alpha W^2 + CW - \delta^2) = 4(W - W_1)(W - W_2)(W - W_3),$$

where W_1, W_2, W_3 are roots of the cubic polynomial $P(W)$. We then have to distinguish between four different cases:

(i) $P(W)$ has only one root. This is the case when

$$W_1 = \frac{-\alpha}{3} = \left(\frac{C}{3}\right)^{1/2} = \delta^{2/3}.$$

The solution is

$$\begin{aligned} &M = \left(\frac{2\varepsilon}{3a_0} + \frac{\lambda_0}{\tau}\right)^{1/2}, \qquad \tau^2 = c_1 y + c_2, \qquad c_1^2 = 8a_0\lambda_0, \\ &\Phi = \frac{3\delta}{4\alpha}\tau^2 - \frac{9\delta}{2\alpha^2}\tau + \frac{27\delta}{2\alpha^3}\ln(3 + \alpha\tau) + \Phi_0, \qquad \psi = M\exp[\mathrm{i}(\varepsilon t + \Phi)]. \end{aligned} \tag{4.32}$$

(ii) $P(W)$ has a double root, say $W_1 = W_2 \neq W_3$ and write $P(W)$ as

$$P(W) = 4(W^3 + \alpha W^2 + CW - \delta^2) = 4(W - W_1)^2(W - W_3).$$

We find W_1 and W_3 from the system

$$-\alpha = W_3 + 2W_1, \qquad C = 2W_1W_3 + W_1^2, \qquad \delta^2 = W_3W_1^2.$$

The roots are ordered as follows:

$$\text{(a) } W_1 > \delta^{2/3} \quad \Rightarrow \quad 0 < W_3 < W_1, \qquad \text{(b) } W_1 < \delta^{2/3} \quad \Rightarrow \quad W_1 < W_3.$$

For the case (a) we integrate on $W > W_1$ and $W_3 < W < W_1$, then find

$$W = W_1 + (W_1 - W_3)\,\mathrm{cosech}^2\tau \qquad \text{and} \qquad W = W_1 + (W_3 - W_1)\,\mathrm{sech}^2\tau,$$

respectively. The corresponding solutions are

$$\begin{aligned} &M = (\lambda_0 W)^{1/2}, \qquad \tau = c_1 y + c_2, \qquad c_1^2 = \frac{a_0\lambda_0}{2}(W_1 - W_3), \\ &\Phi = \frac{\delta}{\sqrt{W_1 - W_3}}\int \frac{\mathrm{d}\tau}{W} + \Phi_0, \qquad \psi = M\exp[\mathrm{i}(\varepsilon t + \Phi)]. \end{aligned} \tag{4.33}$$

For the case (b), we integrate on $W > W_3$ and obtain the solution

$$\begin{aligned} &W = W_1 + (W_3 - W_1)\sec^2\tau, \qquad \tau = c_1 y + c_2, \\ &M = (\lambda_0 W)^{1/2}, \qquad c_1^2 = \frac{a_0\lambda_0}{2}(W_3 - W_1), \\ &\Phi = \frac{\delta}{\sqrt{W_3 - W_1}}\int \frac{\mathrm{d}\tau}{W} + \Phi_0, \qquad \psi = M\exp[\mathrm{i}(\varepsilon t + \Phi)]. \end{aligned} \tag{4.34}$$

(iii) $P(W)$ has three distinct roots. In this case, roots $W_{1,2,3}$ are determined by the system

$$W_1 + W_2 + W_3 = -\alpha, \qquad W_1W_2 + W_1W_3 + W_2W_3 = C, \qquad W_1W_2W_3 = \delta^2.$$

We see that two of the roots have the same sign, while the third is always positive. There are two cases to consider:

$$\text{(a) } W_1 < W_2 < 0 < W_3, \qquad \text{(b) } 0 < W_1 < W_2 < W_3, \qquad W_2^2 < \frac{\delta^2}{W_1}.$$

Integration for $W > W_3$ gives for both (a) and (b)

$$W = W_2 + (W_3 - W_2)[\mathrm{cn}(\tau, k)]^{-2}, \qquad \tau = c_1 y + c_2, \qquad k^2 = \frac{W_2 - W_1}{W_3 - W_1},$$
$$M = (\lambda_0 W)^{1/2}, \qquad c_1^2 = \frac{a_0\lambda_0}{2}(W_3 - W_1), \tag{4.35}$$
$$\Phi = \frac{\delta}{\sqrt{W_3 - W_1}} \int \frac{\mathrm{d}\tau}{W} + \Phi_0, \qquad \psi = M \exp[\mathrm{i}(\varepsilon t + \Phi)]$$

with $a_0, \lambda_0 > 0$. Integrating on $W_1 < W < W_2$ we get

$$W = W_1 + (W_2 - W_1)\mathrm{sn}^2(\tau, k), \qquad \tau = c_1 y + c_2, \qquad k^2 = \frac{W_2 - W_1}{W_3 - W_1},$$
$$M = (\lambda_0 W)^{1/2}, \qquad c_1^2 = \frac{a_0\lambda_0}{2}(W_3 - W_1),$$
$$\Phi = \frac{\delta}{\sqrt{W_3 - W_1}} \int \frac{\mathrm{d}\tau}{W} + \Phi_0, \qquad \psi = M \exp[\mathrm{i}(\varepsilon t + \Phi)]$$

with $a_0, \lambda_0 < 0$ for the case (a) and $a_0, \lambda_0 > 0$ for the case (b).

(iv) Last we investigate the case where $P(W)$ has two complex roots and a real root. Let $W_2 = p + \mathrm{iq}$, $\mathrm{W}_3 = \mathrm{p} - \mathrm{iq}$, then p, q and W_1 are found from the system of equations

$$2p + W_1 = \alpha, \qquad \frac{\delta^2}{W_1} + 2pW_1 = C, \qquad p^2 + q^2 = \frac{\delta^2}{W_1}.$$

Since $W_1 > 0$, integration on $W > W_1$ gives the following solution for $a_0, \lambda_0 > 0$.

$$W = \frac{A + W_1 + (W_1 - A)\mathrm{cn}(\tau, k)}{1 + \mathrm{cn}(\tau, k)}, \qquad \tau = c_1 y + c_2,$$
$$M = (\lambda_0 W)^{1/2}, \qquad k^2 = \frac{A + p - W_1}{2A}, \tag{4.36}$$
$$\Phi = \frac{\delta}{2\sqrt{A}} \int \frac{\mathrm{d}\tau}{W} + \Phi_0, \qquad c_1^2 = 2Aa_0\lambda_0,$$
$$\psi = M \exp[\mathrm{i}(\varepsilon t + \Phi)], \qquad A^2 = (p - W_1)^2 + q^2.$$

4.6.2. Equation (3.13B) with $C_0 = 0$. In this case, equation for W reduces to

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + \left(\frac{2a_0\lambda_0}{\eta_0^2}\right) W^2 + \frac{2\varepsilon}{\eta_0^2} W. \tag{4.37}$$

We are going to investigate the solutions of equation (4.37) for $\varepsilon = 1$ and $\varepsilon = -1$ separately. By the choice $\eta_0^2 = \frac{a_0\lambda_0}{2} = 1$ this equation transforms to

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + 4W^2 + 2\varepsilon W,$$

which is PXIX of [11] for $\varepsilon = 1$. The first integral is again the elliptic function equation

$$\dot{W}^2 = 4W(C + \varepsilon W + W^2), \qquad C = \mathrm{const}. \tag{4.38}$$

If we label the right-hand side of equation (4.38) as $P(W)$ we find the roots of the cubic polynomial $P(W)$ to be

$$W_1 = \frac{-1 - \sqrt{1 - 4C}}{2}, \qquad W_2 = 0, \qquad W_3 = \frac{-1 + \sqrt{1 - 4C}}{2} \qquad \text{for} \quad \varepsilon = 1,$$

and

$$W_4 = \frac{1 - \sqrt{1 - 4C}}{2}, \qquad W_5 = 0, \qquad W_6 = \frac{1 + \sqrt{1 - 4C}}{2} \qquad \text{for} \quad \varepsilon = -1.$$

We see that ordering of the roots depends on C:

$$\begin{aligned} &C < 0 \quad \Rightarrow \quad W_{1,4} < 0 < W_{3,6} \qquad \text{for} \quad \varepsilon = 1, \\ &0 < C < \tfrac{1}{4} \quad \Rightarrow \quad 0 < W_{1,4} < W_{3,6} \qquad \text{for} \quad \varepsilon = -1. \end{aligned} \tag{4.39}$$

Depending on the integration constant C, we obtain the following solutions for (1.1):

(i) $C = 0$. In this case the solution is given below with $\kappa = \pm 1$ for a suitable sign of a_0.

$$\begin{aligned} &0 < W, && \psi = \left(\frac{2}{a_0}\right)^{1/2} \operatorname{cosech}(\kappa y + c_1) \exp[\mathrm{i}(t + \Phi_0)]; \\ &-1 < W < 0, && \psi = \left(\frac{-2}{a_0}\right)^{1/2} \operatorname{sech}(\kappa y + c_1) \exp[\mathrm{i}(t + \Phi_0)]; \\ &1 < W, && \psi = \left(\frac{2}{a_0}\right)^{1/2} \sec(\kappa y + c_1) \exp[\mathrm{i}(-t + \Phi_0)]. \end{aligned} \tag{4.40}$$

(ii) $C = 1/4$.

$$\begin{aligned} &0 < W, && \psi = \left(\frac{1}{a_0}\right)^{1/2} \tan\left(\frac{\kappa}{\sqrt{2}} y + c_1\right) \exp[\mathrm{i}(t + \Phi_0)]; \\ &0 < W < \frac{1}{2}, && \psi = \left(\frac{1}{a_0}\right)^{1/2} \tanh\left(\frac{\kappa}{\sqrt{2}} y + c_1\right) \exp[\mathrm{i}(-t + \Phi_0)]; \\ &\frac{1}{2} < W, && \psi = \left(\frac{1}{a_0}\right)^{1/2} \coth\left(\frac{\kappa}{\sqrt{2}} y + c_1\right) \exp[\mathrm{i}(-t + \Phi_0)] \end{aligned} \tag{4.41}$$

with $a_0 > 0$, $\kappa = \pm 1$.

(iii) $C < 0$. For both $\varepsilon = \pm 1$, the roots of $P(W)$ have opposite signs. An integration with respect to $W_{3,6} < W$ and $W_{1,4} < W < W_{2,5}$ results in the following solutions:

$$\begin{aligned} &\psi = \left(\frac{\sqrt{1-4C} - \varepsilon}{a_0}\right)^{1/2} [\operatorname{cn}(c_1 y + c_2, k)]^{-1} \exp[\mathrm{i}(\varepsilon t + \Phi_0)], && a_0 > 0, \\ &\psi = \left(\frac{\sqrt{1-4C} + \varepsilon}{-a_0}\right)^{1/2} \operatorname{cn}(c_1 y + c_2, k) \exp[\mathrm{i}(\varepsilon t + \Phi_0)], && a_0 < 0 \end{aligned} \tag{4.42}$$

with $c_1^2 = \sqrt{1-4C}$, $k^2 = \frac{\varepsilon + \sqrt{1-4C}}{2\sqrt{1-4C}}$.

(iv) $0 < C < \frac{1}{4}$. We list the solutions following the interval that the integration has been taken on:

$$\begin{aligned} &W_2 < W, \\ &\psi = c_1 \left(\frac{2}{a_0}\right)^{1/2} \left(\frac{1 - \sqrt{1-4C}}{1 + \sqrt{1-4C}}\right)^{1/2} \operatorname{tn}(c_1 y + c_2, k) \exp[\mathrm{i}(t + \Phi_0)], \qquad a_0 > 0; \\ &W_1 < W < W_3, \\ &\psi = c_1 \left(\frac{-2}{a_0}\right)^{1/2} \left[\frac{1 - \sqrt{1-4C}}{1 + \sqrt{1-4C}} + \operatorname{cn}^2(c_1 y + c_2, k)\right]^{1/2} \exp[\mathrm{i}(t + \Phi_0)], \qquad a_0 < 0 \end{aligned} \tag{4.43}$$

with $c_1^2 = \frac{1+\sqrt{1-4C}}{2}$, $k^2 = \frac{2\sqrt{1-4C}}{1+\sqrt{1-4C}}$.

$W_6 < W$,

$$\psi = \left(\frac{1-\sqrt{1-4C}}{a_0}\right)^{1/2}\left[1+\frac{2\sqrt{1-4C}}{1-\sqrt{1-4C}}(\mathrm{cn}(c_1 y + c_2, k))^{-2}\right]^{1/2}\exp[\mathrm{i}(-t+\Phi_0)];$$

$W_5 < W < W_4$, (4.44)

$$\psi = \left(\frac{1-\sqrt{1-4C}}{a_0}\right)^{1/2}\mathrm{sn}(c_1 y + c_2, k)\exp[\mathrm{i}(-t+\Phi_0)]$$

with $a_0 > 0$, $c_1^2 = \frac{1+\sqrt{1-4C}}{2}$, $k^2 = \frac{1-\sqrt{1-4C}}{1+\sqrt{1-4C}}$.

(v) $\frac{1}{4} < C$. We have two solutions for $a_0 > 0$:

$$\psi = \left(\frac{2}{a_0}\right)^{1/2} C^{1/4}\,\mathrm{tn}(C^{1/4}y + c_1, k)\,\mathrm{dn}(C^{1/4}y + c_1, k)\exp[\mathrm{i}(\varepsilon t + \Phi_0)] \quad (4.45)$$

with $k^2 = \frac{2C^{1/2}-\varepsilon}{4C^{1/2}}$.

4.7. Solutions using the subalgebra $L_{2,7}$

This subalgebra leads to second-order nonlinear ODEs of which solutions are written in terms of the Painlevé transcendents.

(i) Equation (3.15*b*) with $C_0 \neq 0$. If we set $\lambda = \lambda_0$, $\eta = \eta_0\xi$, this equation transforms to

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + \left(\frac{2a_0\lambda_0}{\eta_0^2}\right)W^2 + \frac{\eta}{\eta_0^3}W + 2\left(\frac{C_0}{\lambda_0\eta_0}\right)^2 W^{-1}.$$

For the choice $\eta_0 = -1$ this equation becomes

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + 4\gamma W^2 - \eta W + 2\delta^2 W^{-1},$$

where $\gamma = a_0\lambda_0/2$ and $\delta^2 = (C_0/\lambda_0)^2$. This equation is quite similar to equation PXXXIV of [11] and is integrated in the same way. We put

$$2\gamma W = \dot{V} + V^2 + \frac{\eta}{2}$$

and see that V is solved by the equation

$$\ddot{V} = 2V^3 + \eta V + k, \qquad k = -\tfrac{1}{2} \pm 4a_0\delta\mathrm{i},$$

which is the second Painlevé transcendent. Hence we find $V = P_{\mathrm{II}}(-\xi)$. Since W is complex-valued, λ_0 is a complex constant that must be chosen appropriately to make $\lambda_0 W$ real.

(ii) Equation (3.15*b*) with $C_0 = 0$. Under the same transformation $\lambda = \lambda_0$, $\eta = \eta_0\xi$, this equation reduces to

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + \left(\frac{2a_0\lambda_0}{\eta_0^2}\right)W^2 + \frac{\eta}{\eta_0^3}W.$$

With the choice $\eta_0 = 2^{-1/3}$ and $\lambda_0 = 2^{1/3}/a_0$ it is nothing but the canonical equation PXX of [11]:

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + 4W^2 + 2\eta W.$$

If we set $U^2 = W$, the second Painlevé transcendent P_{II} makes its appearance again

$$\ddot{U} = 2U^3 + \eta U.$$

We can write the solution as

$$\psi = \frac{2^{1/6}}{a_0^{1/2}} P_{\mathrm{II}}(2^{-1/3}\xi) \exp\left[\mathrm{i}\left(\frac{xt}{2} - \frac{t^3}{6} + \Phi_0\right)\right]. \tag{4.46}$$

4.8. Solutions using the subalgebra $L_{2,10}$

Equation (4.5) for (3.21*b*) is

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 - \frac{1}{\dot{\eta}}\left(\frac{\ddot{\eta}}{\dot{\eta}} + \frac{\dot{\lambda}}{\lambda}\right)\dot{W} + \frac{2a_0\lambda}{\dot{\eta}^2}W^2 + 2\left(\frac{C_0}{\lambda\dot{\eta}}\right)^2 W^{-1}$$
$$+ \frac{1}{\dot{\eta}^2}\left(2\xi^2 - 2a\varepsilon + \frac{\dot{\lambda}^2}{2\lambda^2} - \frac{\ddot{\lambda}}{\lambda}\right)W. \tag{4.47}$$

Just as for the subalgebra $L_{2,1}$, this equation seems a candidate to be transformable to PXXXIII or PXXXIV. By comparison of the coefficients of W^2 and $\dot{W}$ in (4.47) with (4.7) and (4.8) it turns out that λ and η must be chosen to be

$$\lambda = \lambda_0, \qquad \eta = \eta_0\xi.$$

For this choice of λ and η (4.47) becomes

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + \frac{2a_0\lambda_0}{\eta_0^2}W^2 + 2\left(\frac{C_0}{\lambda_0\eta_0}\right)^2 W^{-1} + \left(\frac{2}{\eta_0^4}\eta^2 - \frac{2a\varepsilon}{\eta_0^2}\right)W. \tag{4.48}$$

Since the coefficient of W includes the term η^2, equation (3.21*b*) cannot be transformed to one of the canonical forms; thus it does not have the Painlevé property.

If we consider equation (3.21*b*) with $C_0 = 0$, the same arguments above apply. We finally state that equation (3.21*b*) admits no point symmetries and no constant solutions.

4.9. Solutions using the subalgebra $L_{2,12}$

Singularity analysis and integration of equation (3.25*b*) are performed in [3] in which the authors showed that equation (3.25*b*) for $C_0 = 0$, namely

$$2\dot{\chi}\dddot{\chi} - \ddot{\chi}^2 + \left(\frac{\xi^2}{4} - 2a\right)\dot{\chi}^2 - \frac{1}{4}\chi^2 - 4a_0\dot{\chi}^3 = 0,$$

is of the Painlevé type and allows a first integral of the form

$$\ddot{\chi}^2 = -\tfrac{1}{4}(\chi - \dot{\chi}\xi)^2 + 2a\dot{\chi}^2 + 2a_0 + \dot{\chi}^3 + C\dot{\chi}. \tag{4.49}$$

They also showed that the solution of (4.49) is given by

$$\chi(\xi) = -\frac{k}{a_0}\left[\frac{\dot{W}^2}{4W} - \frac{W^3}{4} - zW^2 + (1 - z^2 + \alpha)W + \frac{\beta}{2W} + \frac{2}{3}(\alpha + 1)z + \frac{az}{3k^2}\right], \tag{4.50}$$

where $W(z)$ satisfies the Painlevé transcendent P_{IV}:

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + \frac{3}{2}W^3 + 4zW^2 + 2(z^2 - \alpha)W + \frac{\beta}{W}, \tag{4.51}$$

where $z = k\xi$, $k^4 = -1/16$ and the constants α, β are determined by

$$-2(\alpha + 1)^2 + 3\beta = 4(2a^2 - 3Ca_0),$$
$$(\alpha + 1)[2(\alpha + 1)^2 + 9\beta] = (a/k^2)(4a^2 - 9Ca_0).$$

Table 3. Three-dimensional subalgebras and invariant solutions ($\varepsilon = \pm 1$).

No	Basis	Solution
$L_{3,1}$	$T + \varepsilon E, P_1, P_2$	$\psi = (\varepsilon/a_0)^{1/2} \exp[\mathrm{i}(\varepsilon t + \Phi_0)]$
$L_{3,2}$	$J + aE, D + bE, T$	$\psi = [a(x^2 + y^2)]^{-1/2} \exp[\mathrm{i}(a_0 \tan^{-1}(y/x) + \Phi_0)]$
$L_{3,3}$	$D + aE, T, P_1$	$\psi = (\sqrt{2}/y) \exp(\mathrm{i}\Phi_0)$
$L_{3,4}$	$J + \varepsilon T + aE, P_1, P_2$	$\psi = (a/\varepsilon)^{1/2} \exp[\mathrm{i}(at/\varepsilon + \Phi_0)]$

To sum up, the solution to (1.1) is given by formula (3.24) with

$$M = \sqrt{\dot{\chi}}, \qquad \Phi = \int \frac{\xi(1 + \dot{\chi})}{4\dot{\chi}} \, \mathrm{d}\xi.$$

Remark. We have not been able to successfully treat the reduced equation corresponding to the subalgebra $L_{2,11}$.

4.10. Invariant solutions from three-dimensional subalgebras

We conclude by looking for further solutions on the basis of three-dimensional subalgebras. Invariance under those subalgebras gives rise to algebraic equations only rather than ODEs. We simply list nontrivial solutions in table 3, ruling out those leading to trivial ones.

5. Concluding remarks

In this paper we considered a cubic Schrödinger equation in (2+1) dimension and gave a comprehensive investigation of group-invariant solutions by performing reductions to ODEs using appropriate subgroups of the symmetry group and by solving them whenever possible. Our approach is a powerful combination of the group theory and singularity analysis. In this respect, it is very similar to that used in a series of papers for an investigation of group-invariant solutions of a (3+1)-dimensional generalized nonlinear Schrödinger equation (which, as special cases, contains dissipative, cubic and quintic Schrödinger equations) in [1–3]. The solutions obtained represent a conjugacy class of solutions. The entire class can be produced by applying a general symmetry group transformation (2.10)–(2.11) to the representative solution. The solutions include Jacobi elliptic functions, their degenerate forms as elementary ones and Painlevé irreducibles. What emerges from a comparison of results in 3+1 and 2+1 dimensions is that there are reductions to ODEs with similar structure in both dimensions (for instance, Painlevé transcendents are ubiquitous), while the corresponding invariant solutions have truly different expressions.

We mention that some similarity solutions of (1.1) have already been discussed in [14] in which the author merely chooses an arbitrary combination of the generators for performing a symmetry reduction. Reduction to an ODE is achieved in two successive steps, first to a (1+1)-dimensional PDE and then to an ODE by imposing an inherited symmetry of the reduced PDE. In contrast, in the present paper we classify solutions into a list of conjugacy classes by using a rigorous subgroup classification so that every solution is conjugate to precisely one in the list.

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